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Diversities and Conformities

1 Definition

Definition 1. Let (X, δ) be a diversity, $\{x_n\}$ a sequence of elements of X. We say that $\{x_n\}$ *converges to a limit* x if

$$\lim_{N\to\infty} \sup_{i_1,\ldots,i_n \geq N} \delta(\{x, x_{i_1}, \ldots, x_{i_n}\}) = 0$$

Definition 2. Let (X, δ) be a diversity, $\{x_n\}$ a sequence of elements of X. We say that $\{x_n\}$ is a *Cauchy sequence* if

$$\lim_{N\to\infty} \sup_{i_1,\ldots,i_n \geq N} \delta(\{x_{i_1}, \ldots, x_{i_n}\}) = 0$$

Definition 3. We call a diversity *complete* if every Cauchy sequence converges.

Sanity Check 1. Every convergent sequence is Cauchy.

Proof. Monotonicity. □

2 Metrics and Diversities

Theorem 1. Let (X, δ) be a diversity, d its induced metric. If (X, d) is a complete metric space, then (X, δ) is a complete diversity.

Proof. Suppose that (X, d) is complete. Let $\{x_n\}$ be a Cauchy sequence in (X, δ). Then it is also Cauchy in (X, d), and therefore converges to some element x. We claim that $x_n \to x$ in (X, δ). To this end, let $\epsilon > 0$. Then there exists N such that:

- $d(x_n, x) < \epsilon$ for all $n > N$ (since $x_n \to x$ in (X, d))
- $\delta(\{x_{n_1}, x_{n_2}, \ldots, x_{n_m}\}) < \epsilon$ for all $n_i > N$ (since $\{x_n\}$ is Cauchy in (X, δ)).

Therefore, for all $n_1, \ldots, n_m > N$,

$$\begin{aligned} \delta(\{x, x_{n_1}, \ldots, x_{n_m}\}) &\leq \delta(\{x, x_{n_1}\}) + \delta(\{x_{n_1}, \ldots, x_{n_m}\}) \\ &= d(x, x_{n_1}) + \delta(\{x_{n_1}, \ldots, x_{n_m}\}) < 2\epsilon \end{aligned}$$

i.e., $x_n \to x$ in (X, δ). □

Lemma 1. Let (X, d) be a metric space, $\{x_n\}$ a Cauchy sequence in X. If there exists some subsequence $\{x_{i_n}\}$ such that $x_{i_n} \to x$, then $x_n \to x$.

Proof. Let $\epsilon > 0$. Then $d(x_n, x) \leq d(x_n, x_{i_m}) + d(x_{i_m}, x) < 2\epsilon$ for m, n large enough. □

Lemma 2. Let (X, δ) be a diversity, d its induced metric. Let $\{x_n\}$ be Cauchy in (X, d). Then it has a subsequence that is Cauchy in (X, δ).

Proof. Define the subsequence $\{x_{n_i}\}$ by

$$n_i = \min\{n : d(x_n, x_m) < 2^{-i} \text{ for all } m \geq n\}$$

Then given $\epsilon > 0$, choose N such that $2^{1-N} < \epsilon$. Then for all $i_1 \leq i_2 \leq \cdots \leq i_m$ greater than N,

$$\begin{aligned}
\delta(\{x_{i_1}, \ldots, x_{i_m}\}) &\leq \delta(\{x_{i_1}, x_{i_2}\}) + \cdots + \delta(\{x_{i_{m-1}}, x_{i_m}\}) \\
&< 1/2^{i_1} + \cdots + 1/2^{i_m} \\
&< \sum_{i=N}^{\infty} 1/2^i \\
&= 2^{1-N} < \epsilon
\end{aligned}$$

That is, $\{x_{n_i}\}$ is Cauchy in (X, δ). □

Theorem 2. Let (X, δ) be a diversity, d its induced metric. If (X, δ) is a complete diversity, then (X, d) is a complete metric space.

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X, d). Then by Lemma 2 it has a subsequence that is Cauchy in (X, δ), which converges to some element x since the diversity is complete. (It converges in both (X, δ) and (X, d).)

Then by Lemma 1, $x_n \to x$ in the metric; i.e., $\{x_n\}$ converges. □

Putting these two theorems together, we get

Corollary 1. A diversity (X, δ) is complete iff its induced metric is complete.

However, we do not have equivalence of Cauchiness, as we will see.

Theorem 3. There exists a diversity (X, δ, d), and a sequence $\{x_n\}$ in X, which is Cauchy in (X, d) but not in (X, δ).

Proof. Let (X, δ) be the Steiner tree diversity on $\mathbb{R}^2$. Define the sets S_i, $i \geq 2$ by

$$S_i = \left\{ \left(\frac{n + 1/2}{i^2}, \frac{m + 1/2}{i^2} \right) : 0 \leq n \leq i, 0 \leq m \leq i \right\}$$

That is, we have a sequence of $i \times i$ lattices on the square $[0, 1/i] \times [0, 1/i]$, and the points in S_i are the midpoints.

Order the points in each S_i somehow, and define the sequence

$$\{x_n\} = S_2 S_3 S_4 \cdots$$

Then $\{x_n\}$ is Cauchy in (X, d), since eventually every pair of points is confined to the square $[0, \epsilon] \times [0, \epsilon]$. However, it is not Cauchy in (X, δ), for the following reason:

Consider a minimum spanning tree connecting all the points of some S_i. Since there are i^2 points in S_i, we must have at least $i^2 - 1$ edges. Each edge must have length $\geq 1/i^2$, since that is the minimum spacing between points of S_i. So the total length of the edges is

$$\frac{1}{i^2} \cdot (i^2 - 1) = 1 - \frac{1}{i^2} \geq \frac{3}{4}$$

for every $i \geq 2$. Thus for any $N > 0$,

$$\sup_{i_1, \ldots, i_n \geq N} \delta(\{x_{i_1}, \ldots, x_{i_n}\}) \geq \frac{3}{4} C$$

where C is a constant comparing the length of a minimum spanning tree to that of a minimum Steiner tree, in the plane. □

3 Completion

Theorem 4. Every diversity (X, d) can be embedded in a complete diversity.

Proof. Let $\hat{X}$ be the set of all Cauchy sequences in X. Identify any two sequences $\{x_i\}, \{y_i\}$ which satisfy $\lim_{n\to\infty} \delta(\{x_n, y_n\}) = 0$ (so $\hat{X}$ is actually a set of equivalence classes). Define the function $\hat{\delta}$ from $\mathcal{P}_{\text{fin}}(\hat{X}) \to \mathbb{R}$ by

$$\hat{\delta}(\{\{x_i^1\}, \{x_i^2\}, \ldots, \{x_i^n\}\}) = \lim_{N\to\infty} \sup_{i_1, \ldots, i_n \geq N} \delta(\{x_{i_i}^1, x_{i_2}^2, \ldots, x_{i_n}^n\})$$

It can then be shown that $(\hat{X}, \hat{\delta})$ is a complete diversity, and that (X, δ) can be embedded isodiversically in it by $x \mapsto \{x, x, x, \ldots\}$. The proof is straightforward but tedious. □

4 Some Examples

Example 1. *Let (X, d) be a complete metric space, (X, δ) its diameter diversity. Then (X, δ) is complete.*

Example 2. *The Steiner tree diversity on $\mathbb{R}^2$ is complete. (Its induced metric is the Euclidean one, which is complete.)*

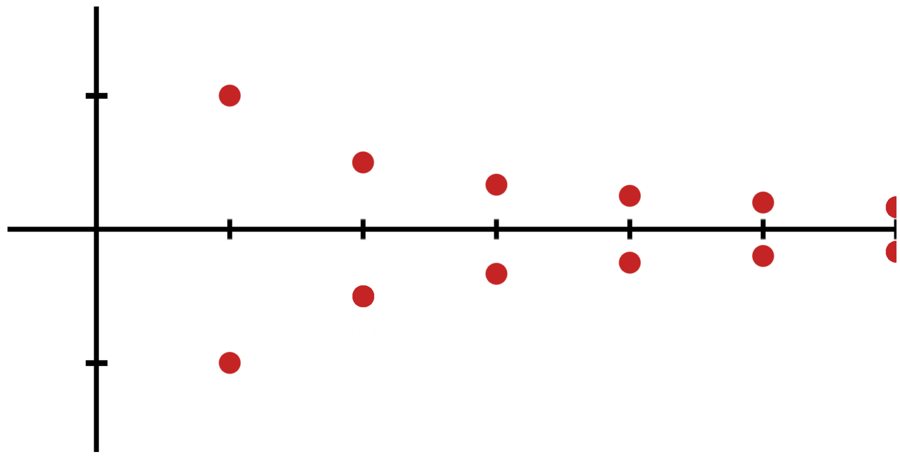
Example 3. *Where diversities behave very differently from their induced metric, the reason is that we can find sets A such that $\delta(A)$ is very large, while $\delta(\{a, b\})$ is small for any $a, b \in A$. We might ask how extreme this effect can be.*

Specifically, does a diversity (X, δ) exist such that?:

- *For all $\epsilon > 0$, there exists $x, y \in X$ such that $\delta(\{x, y\}) < \epsilon$.*
- *There exists some constant C such that for all $Z \subset X$, $\delta(Z) \geq C|Z|$.*

Yes. *Consider the euclidean diameter diversity on the set*

$$X = \{(n, 1/n) : n \in \mathbb{N}\} \cup \{(n, -1/n) : n \in \mathbb{N}\}$$



For any $\epsilon > 0$, *we can find* $n > \frac{1}{2\epsilon}$, *and the points* $(n, 1/n)$ *and* $(n, -1/n)$ *will be within* ϵ *of each other. However, it is clear that any finite set* A *will have diameter* $\geq |A|/2$.

However, this pathology will not affect our results on convergence, since no such diversity can have a limit point (defined as a point x, *with a sequence* $\{x_n\}$ *such that* $x_i \neq x$ *for all* i, *but* $x_n \to x$). *The reason is simple: for a sequence* $\{x_n\}$ *to converge in this diversity, it would have to be eventually constant. So "*$x_i \neq x$ *for every* i*" is incompatible with "*$x_n \to x$*".*

5 Fixed Points

Definition 4. Let (X, δ) be a diversity, $T : X \to X$ a function such that

$$\delta(T(A)) \leq k\delta(A)$$

for all finite $A \subset X$, some $k \in (0, 1)$. We call T a *contraction mapping* with *Lipschitz constant* k.

Theorem 5. Let (X, δ) be a complete diversity, $T : X \to X$ a contraction mapping with Lipschitz constant k. Then there exists a unique point $x_0 \in X$ such that $T^n(x) \to x_0$ as $n \to \infty$, for all $x \in X$.

Proof. Let $N \in \mathbb{N}$. Then for all $i_1, i_2, \ldots, i_n > N$,

$$\begin{aligned}
&\delta(\{T^{N+i_1}(x), T^{N+i_2}(x), \ldots, T^{N+i_n}(x)\}) \\
&\qquad \leq \delta(\{T^{N+i_1}(x), T^{N+i_2}(x)\}) + \cdots + \delta(\{T^{N+i_{n-1}}(x), T^{N+i_n}(x)\}) \\
&\qquad \leq (k^{N+i_1} + k^{N+i_2} + \cdots + k^{N+i_n})\delta(\{x, T(x)\}) \\
&\qquad \leq \frac{k^N}{1-k}\delta(\{x, T(x)\})
\end{aligned}$$

So by taking N large enough, we can force $\delta(\{T^{N+i_1}(x), \ldots, T^{N+i_n}(x)\})$ as small as we like, and the sequence is Cauchy. Then since (X, δ) is complete, it converges to some fixed point x_0.

As for uniqueness, let y_0 be another fixed point. Then

$$\delta(\{x_0, y_0\}) = \delta(\{T(x_0), T(y_0)\}) \leq k\delta(\{x_0, y_0\})$$

So $\delta(\{x_0, y_0\}) = 0$. □

6 Uniform Spaces and Conformities

Definition 5. Let X be a set, $\mathcal{F}$ a collection of subsets of $\mathcal{P}(X)$. We call $\mathcal{F}$ a *filter* if

- Whenever $X, Y \in \mathcal{F}$, so is $X \cap Y$.
- Whenever $X \in \mathcal{F}$, $Z \supseteq X$, $Z \in \mathcal{F}$.

If $\varnothing \notin \mathcal{F}$, we call $\mathcal{F}$ a *proper filter*.

Definition 6. Let X be a set, $\mathcal{F}$ a collection of subsets of $\mathcal{P}(X)$. We call $\mathcal{F}$ a *filter base* if $\mathcal{F}$ becomes a filter by adding supersets of its elements.

Definition 7. Let X be a set, $\mathcal{U}$ be a collection of subsets of $\mathcal{P}_{\text{fin}}(X)$. We call $\mathcal{U}$ a *conformity* if

- $\mathcal{U}$ is a filter on $\mathcal{P}_{\text{fin}}(X)$.
- For all $U \in \mathcal{U}$, all $x \in X$, $\{x\} \in U$.
- If $a \in U$, $b \subseteq a$, then $b \in U$.
- For all $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that

$$V \circ V := \{u \cup v : u, v \in V \text{ and } u \cap v \neq \varnothing\} \subseteq U$$

The pair $(X, \mathcal{U})$ is called a *uniform space*.

This definition corresponds to that of a *uniformity* from metric space theory. The name *conformity* comes from the question, "how does one make a uniformity from a diversity?".

Definition 8. By a *base* for a conformity, we mean a filter base.

Note. Composition, as defined above, is commutative, so that $V \circ V \circ V$ is unambiguous.

Lemma 3. Let $(X, \mathcal{U})$ be a conformity. Then for any $U \in \mathcal{U}$, $n \in \mathbb{N}$, there exists $V \in \mathcal{U}$ such that

$$V \circ V \circ V \subseteq U$$

Proof. Choose V' so that $V' \circ V' \subseteq U$, then V such that $V \circ V \subseteq V'$. Then

$$V \circ V \circ V \subseteq (V \circ V) \circ (V \circ V) = V' \circ V' \subseteq U$$

□

There is a natural way to create conformities from diversities; let (X, δ) be a diversity. Then let

$$\mathcal{U}_{\text{base}} = \{\{x \subset X : |x| < \infty \text{ and } \delta(X) < \epsilon\} : \epsilon > 0\}$$

It is then easily verified that

$$\mathcal{U} = \{U : \exists V \in \mathcal{U}_{\text{base}} \text{ such that } V \subseteq U\}$$

is a diversity. (The last axiom is satisfied by the triangle inequality.)

We can also create conformities from multiple diversities; the generated filter is guaranteed to be proper since every base element contains the diagonal. (So every pair of base elements will have a nonempty intersection.)

6.1 Uniform Continuity

Definition 9. Let (X, δ_X), (Y, δ_Y) be diversities, $f : X \to Y$. We say f is *continuous at a point* x if for all $\epsilon > 0$, there exists some $d = d(x) > 0$ such that whenever $x \in A$, $\delta_X(A) < d$, then $\delta_Y(f(A)) < \epsilon$.

If f is continuous at every point $x \in \Omega$, we say f is *continuous on* Ω.

Definition 10. We say f is *uniformly continuous* if for all $\epsilon > 0$, there exists $d > 0$ such that $\delta_X(A) < d \implies \delta_Y(f(A)) < \epsilon$.

Sanity Check 2. Every uniformly continuous function is continuous.

Theorem 6. Let (X, δ_X), (Y, δ_Y) be diversities, $\mathcal{U}_X$ and $\mathcal{U}_Y$ their conformities. Let $f : X \to Y$. Then f is uniformly continuous iff for all $U \in \mathcal{U}_Y$, $\{f^{-1}(u) : u \in U\} \in \mathcal{U}_X$.

This theorem lets us define uniform continuity purely in terms of conformities, without reference to any underlying diversity.

6.2 Power Conformities

Definition 11. Let (X, δ) be a diversity. We define the *power diversity* $(\mathcal{P}_{\text{fin}}(X), \hat{\delta})$ by

$$\hat{\delta}(\{A_1, \ldots, A_n\}) = \sup_{a_i \in A_i} \delta\left(\bigcup_{i=1}^{n} \{a_i\}\right)$$

Theorem 7. The "power diversity" is a diversity.

Proof. This follows easily from the fact that δ is a diversity. □

Example 4. *The power diversity of a discrete diversity is a discrete diversity.*

Example 5. *Let (X, δ) be the diameter diversity of a metric d. Then the power diversity definition becomes*

$$\hat{\delta}(\{A_1, \ldots, A_n\}) = \sup_{i \neq j} d(a_i, a_j)$$

where $a_i \in A_i$, $a_j \in A_j$.

Note. Notice that (X, δ) can be embedded isodiversically in its power diversity by the mapping $x \mapsto \{x\}$. This embedding is never dense.

Definition 12. Let $(X, \mathcal{U})$ be a conformity. Then we define the *power conformity* $\mathcal{U}^P$ as the conformity generated by sets of the form

$$U_v = \{\{A_1, \dots, A_n\} : \{a_1, \dots a_n\} \in v \text{ for any choice of } a_i\text{'s } \}$$

for each $v \in \mathcal{U}$.

Theorem 8. The Steiner tree diversity on $\mathbb{R}^2$ is not uniformly continuous from its power uniformity to the Euclidean diameter uniformity.

Proof. Let $\epsilon > 0$. Let A and B be two clusters of points of diameter $\epsilon/10$, separated from each other but contained in a set of diameter ϵ. Then $\hat{\delta}(A, B) < \epsilon$ but we can let $\delta(B)$ and $\delta(A)$ be as different as we want. □

Note. This strongly suggests that our definition of "power conformity" is bunk. We would like one such that every diversity is uniformly continuous from its own conformity to $\mathbb{R}$. However, this does not seem possible with our current set of conformity axioms:

For a diversity δ to be uniformly continuous from a power conformity to $\mathbb{R}$, the sets $\delta^{-1}([0, \epsilon])$ would need to be members of the conformity, for all $\epsilon > 0$. In other words, the power conformity would be generated by a "power diversity" of the form

$$\hat{\delta}(\{A_1, \dots, A_n\}) = \text{diam}[\delta(A_1), \dots, \delta(A_n)]$$

However, the current set of conformity axioms do not provide enough information to simulate diam without an actual diversity backing it.

Metric spaces would have the same problem, but their analogues to power conformities are product uniformities. And as it turns out, all the useful product metrics are equivalent, so they can use one that is easy to simulate.

For now, we will keep this definition. It works somewhat:

Theorem 9. Let $(X, \mathcal{U})$ be a conformity. If a pseudodiversity δ is uniformly continuous from the power conformity $\mathcal{U}^P$ to the Euclidean diversity on $\mathbb{R}$, then

$$U_\epsilon = \{A \subset \mathcal{P}_{\text{fin}}(X) : \delta(A) < \epsilon\}$$

is a member of $\mathcal{U}$ for all ϵ.

Proof. If δ is uniformly continuous, then $\forall \epsilon > 0$, we have that

$$\{\{A_1, \dots, A_n\} : \text{diam}(\{\delta(A_1), \dots, \delta(A_n)\}) < \epsilon\} \in \mathcal{U}^P \tag{1}$$

Since the sets $\{U_v\}_{v\in\mathcal{U}}$ generate the power conformity, some U_v lives inside the above set. We claim that every element of v lives inside U_v:

Let $A = \{a_1, \ldots, a_n\} \in v$. Then by monotonicity,

$$\{\{a_1\}, \{a_1, a_2\}, \ldots, A\} \in U_v$$

so that by (1),

$$\operatorname{diam}\left(\{\delta(\{a_1\}), \delta(\{a_1, a_2\}), \ldots, \delta(A)\}\right) < \epsilon$$

so that $\delta(A) < \epsilon$; i.e., $A \in U_\epsilon$. We conclude that $v \subseteq U_\epsilon$, so by upper closure $U_\epsilon \in \mathcal{U}$. □

Theorem 10. (Partial converse) Let $(X,\mathcal{U})$ be a conformity. If a *diameter* diversity δ satisfies $U_\epsilon \in \mathcal{U}$ for all $\epsilon > 0$, then δ is uniformly continuous from $\mathcal{P}_{\text{fin}}(X)$ to $\mathbb{R}$.

Proof. Fix $\epsilon > 0$, and let $A_1, \ldots, A_n \in \mathcal{P}_{\text{fin}}(X)$ such that $\hat{\delta}(\{A_1, \ldots, A_n\}) < \epsilon/2$. Then

$$\begin{aligned}
\operatorname{diam}(\{\delta(A_1), \ldots, \delta(A_n)\}) &= |\delta(A_i) - \delta(A_j)| && \text{for some } i, j \\
&= |d(a_u^i, a_v^i) - d(a_p^j, a_q^j)| && \text{for some } u, v, p, q \\
&\leq |d(a_u^i, a_v^i) - d(a_u^i, a_p^j)| + |d(a_u^i, a_p^j) - d(a_p^j, a_q^j)| \\
&\leq d(a_p^j, a_v^i) + d(a_u^i, a_q^j) \\
&< 2(\epsilon/2)
\end{aligned}$$

which completes the proof. □

6.3 Diversification

Lemma 4. Let $(X,\mathcal{U})$ have a countable base. Then it has a countable base $\{U_n\}$ satisfying $U_0 = \mathcal{P}_{\text{fin}}(X)$, $U_i \circ U_i \circ U_i \subseteq U_{i-1}$ for all i.

Proof. Let $\{V_n\}$ be a countable base for $\mathcal{U}$. Define $W_0 = \mathcal{P}_{\text{fin}}(X)$, $W_n = V_n \cap W_{n-1}$. Then $\{W_n\}$ is a nested countable base. Finally, choose $\{U_n\}$ as $U_i = W_{n_i}$, where n_i are chosen inductively as $n_0 = 0$, then $W_{n_i} \circ W_{n_i} \circ W_{n_i} \subseteq W_{n_{i-1}}$. (See Lemma 3.) □

Theorem 11. Let $(X,\mathcal{U})$ be a conformity, $\{U_n\}$ a sequence in $\mathcal{U}$ satisfying $U_0 = \mathcal{P}_{\text{fin}}(X)$, $U_i \circ U_i \circ U_i \subseteq U_{i-1}$ for all $i > 0$. Let $\mathcal{V}$ be the conformity generated by these sets.

Then there exists a pseudodiversity δ on X which generates $\mathcal{V}$.

Proof. We define δ' on $\mathcal{P}_{\text{fin}}$ as follows:

$$\delta'(A) = \begin{cases} 0 & A \in U_n \text{ for all } n \\ 2^{-k} & A \in U_n \text{ for } 0 \leq n \leq k, \text{ but } A \notin U_{k+1} \end{cases}$$

Notice that for $k \geq 0$,

$$\delta'^{-1}([0, 2^{-k}]) = U_k \tag{2}$$

Also note that δ' is monotonic, since if $A \subseteq B$, then A has to be in every U_k that B is. (This is an axiom for $\mathcal{U}$.)

Now, define a **chain** as a finite sequence $\{A_i\}_1^n$ with $A_i \cap A_{i+1} \neq \varnothing$ for $i = 1, \ldots, (n-1)$. Define a **cycle** to be a chain such that $A_1 \cap A_n \neq \varnothing$.

Next, define

$$\bar{\delta}(A) = \inf_{\text{chains covering } A} \sum_{i=1}^{n} \delta'(A_i) \tag{3}$$

$$\delta(A) = \inf_{\text{cycles covering } A} \sum_{i=1}^{n} \delta'(A_i) \tag{4}$$

Also, let $\delta(\varnothing) = \bar{\delta}(\varnothing) = 0$.

There are three stages to our proof:

1. First, we show that δ is a pseudodiversity. We notice that for any singleton $\{x\}$, $\{x\}$ is in every member of $\mathcal{U}$, so that $\delta'(\{x\}) = 0$. Then since $\{x\}$ forms a single-element chain covering itself, $\delta(\{x\}) = 0$.

   Also, the triangle equality holds: let $\epsilon > 0$, $A, C \in \mathcal{P}_{\text{fin}}(X)$ and $B \in \mathcal{P}_{\text{fin}}(X)$ be nonempty. Choose chains $\{A_i\}_1^n$ and $\{B_i\}_1^m$ which cover $A \cup B$ and $B \cup C$, respectively, and for which

   $$\sum_{i=1}^{n} \delta'(A_i) \leq \delta(A \cup B) + \epsilon \qquad \text{and} \qquad \sum_{i=1}^{m} \delta'(B_i) \leq \delta(B \cup C) + \epsilon$$

   Then $\{A_i\}_1^n \cup \{B_i\}_1^m$ forms a chain (after reordering) which covers $A \cup C$, so that

   $$\delta(A \cup C) \leq \sum_{i=1}^{n} \delta'(A_i) + \sum_{i=1}^{m} \delta'(B_i) \leq \delta(A \cup B) + \delta(B \cup C) + 2\epsilon$$

   That ϵ is arbitrary gives the result.

2. Next, we claim that

   $$\delta \leq \bar{\delta} \leq 2\delta \tag{5}$$

   This follows easily since

   - Every cycle is a chain, so $\delta \leq \bar{\delta}$.
   - If $\{A_1, \ldots, A_{n-1}, A_n\}$ is a chain, then $\{A_1, \ldots, A_{n-1}, A_n, A_{n-1}, \ldots, A_1\}$ is a cycle — and the sum of δ' over this cycle is less than twice the sum of δ' over the original chain. It follows that $\bar{\delta} \leq 2\delta$.

3. Finally, we claim that $\bar{\delta}$ satisfies

   $$\frac{1}{2}\delta' \leq \bar{\delta} \leq \delta'$$

which will give us that

$$\delta'^{-1}([0,x]) \subseteq \bar{\delta}^{-1}([0,x]) \subseteq \delta'^{-1}([0,2x]) \tag{6}$$

Putting (2) and (6) together gives us

$$U_k = \delta'^{-1}([0,2^{-k}]) \subseteq \bar{\delta}^{-1}([0,x]) \subseteq \delta'^{-1}([0,2^{-K}]) = U_K$$

for $2^{-k} < x < 2^{-K-1}$. Thus $\bar{\delta}^{-1}([0,x])$ is a base for $\mathcal{U}$, and by (5), so is $\delta^{-1}([0,x])$. In other words, δ generates $\mathcal{V}$.

To prove this last step, we first note that $\bar{\delta} \le \delta'$ trivially. To show $\bar{\delta} \ge \delta'/2$, choose $A \in \mathcal{P}_{\text{fin}}(X)$. Our strategy is to induct on the greatest integer N such that $\bar{\delta}(A) < 2^{-N}$. The case $N = 0$ is trivial, since $\delta' \le 1$, so $\bar{\delta}(A) > 1/2 \ge \delta'(A)/2$. (For the same reason, the case $\bar{\delta}(A) = 1$, which is not covered by the induction, is trivial.)

For $N > 0$, choose $\epsilon \in \big(0, 2^{-N} - \bar{\delta}(A)\big)$ and a chain $\{A_i\}_1^n$ such that

$$\sum_{i=1}^{n} \delta'(A_i) = \bar{\delta}(A) + \epsilon < 2^{-N} \tag{7}$$

If $n = 1$, our sum is simply $\delta'(A_1)$, so we have

$$\delta'(A) \le \delta'(A_1) < 2^{-N} < 2\bar{\delta}(A)$$

Otherwise, there is $k < n$ such that

$$\sum_{i=1}^{k-1} \delta'(A_i) \le \frac{\bar{\delta}(A)}{2} \quad \text{and} \quad \sum_{i=k+1}^{n} \delta'(A_i) \le \frac{\bar{\delta}(A)}{2} \tag{8}$$

Since $\{A_i\}_1^{k-1}$ and $\{A_i\}_{k+1}^n$ are chains whose sum under δ' is less than half that of $\{A_i\}_1^n$, the inductive hypothesis applies to them and we may write

$$\begin{aligned}
\delta'(A_1 \cup \cdots \cup A_{k-1}) &\le 2\bar{\delta}(A_1 \cup \cdots \cup A_{k-1}) && \text{inductive hypothesis} \\
&\le 2\sum_{i=1}^{k-1} \delta'(A_i) && \text{definition of } \bar{\delta} \\
&\le \bar{\delta}(A) && \text{by (8)} \\
&< 2^{-N}
\end{aligned}$$

An identical argument gives that $\delta'(A_{k+1} \cup \cdots \cup A_n) < 2^{-N}$, and $\delta'(A_k) < 2^{-N}$ by (7). So

$$(A_1 \cup \cdots \cup A_{k-1}) \in U_{N+1} \text{ and } A_k \in U_{N+1} \text{ and } (A_{k+1} \cup \cdots \cup A_n) \in U_{N+1}$$

Our double-composition hypothesis gives

$$(A_1 \cup \cdots \cup A_{k-1}) \cup A_k \cup (A_{k+1} \cup \cdots \cup A_n) \in U_N$$

And we are done!:

$$\delta'(A) \leq \delta'(A_1 \cup \cdots \cup A_n) \leq 2^{-N} \leq 2\bar{\delta}(A)$$

□

Corollary 2. If $(X,\mathcal{U})$ has a countable base, then there exists a pseudodiversity δ which generates $\mathcal{U}$.

The converse of the above theorem is trivially true.

Note. The pseudodiversity δ is **not**, in general, uniformly continuous from its own power conformity to $\mathbb{R}$.

Theorem 12. Let $(X,\mathcal{U})$ have a countable base. Then the pseudodiversity δ created in the previous theorem is uniformly continuous from its power conformity to $\mathbb{R}$.